Coherent states, even and odd coherent states in a finite-dimensional Hilbert space and their properties

1998 J. Phys. A: Math. Gen. 31 1307

(http://iopscience.iop.org/0305-4470/31/4/018)

J. Phys. A: Math. Gen. **31** (1998) 1307–1317. Printed in the UK PII: S0305-4470(98)85049-X

Coherent states, even and odd coherent states in a finite-dimensional Hilbert space and their properties

B Roy† and P Roy‡

Physics and Applied Mathematics Unit, Indian Statistical Institute, Calcutta—700035, India

Received 4 July 1997, in final form 10 September 1997

Abstract. We have considered a harmonic oscillator in a finite-dimensional Hilbert space spanned by orthogonal polynomials of a discrete variable and constructed coherent states as well as even and odd coherent states of this oscillator. Various properties such as squeezing and antibunching of these states have also been examined.

1. Introduction

Recently, much attention has been paid to the investigation of quantum systems in finite-dimensional Hilbert space. These quantum systems can be associated with spin systems, several-level atoms in quantum optics, electrons on molecules with a finite number of sites etc. Recently they have also been used for the construction of a hermitian phase operator in quantum mechanics (Pegg–Barnett formalism) [1–4]. Finite-dimensional quantum systems are also of great interest because of their connection to quantum cryptography [5], quantum teleportation and superdense coding [6] and the quantum computation [7].

In the Pegg–Barnett formalism the finite-dimensional states are constructed from the eigenstates of an oscillator in a finite-dimensional Hilbert space. This finite-dimensional oscillator is characterized by the following relations [1–4]:

$$\begin{aligned} &a|0\rangle = a^\dagger|N\rangle = 0 \\ &a|n\rangle = \sqrt{n}|n-1\rangle \\ &a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \\ &[a, a^\dagger] = 1 - (N+1)|N\rangle\langle N| \end{aligned} \tag{1}$$

$(N + 1)$ being the dimension of the Hilbert space.

Subsequently, many authors have studied the construction of coherent states defined in finite-dimensional Hilbert space. One definition was proposed by Buzek *et al* [8] and studied further by Miranowicz *et al* [9, 10]. Another definition was proposed by Kuang *et al* [11]. The various nonclassical properties of these finite-dimensional coherent states were studied by many authors [10, 12] and the various approaches were also compared [13].

At this point we note that the oscillator algebra defined in (1) is not a closed one and also a specific coordinate realization of the various operators appearing in (1) has not been found so far. This has motivated us to search for a suitable algebra which has a specific

† E-mail address: barnana@isical.ernet.in
‡ E-mail address: pinaki@isical.ernet.in

0305-4470/98/041307+11$19.50

coordinate realization and which in the limit of large N becomes the standard harmonic oscillator algebra. In this case we shall use the formalism due to Atakishiev *et al* [14]. It will be seen that the oscillator algebra is related to the $SU(2)$ algebra and can be realized in the space of Kravchuk polynomials (which are orthogonal polynomials of a discrete variable). Coherent states, in the manner of Perelomov [15], will then be constructed and we shall examine various nonclassical properties of these coherent states. Subsequently we shall also construct and study nonclassical properties of the even and odd superpositions of two finite-dimensional coherent states [16]. It will be shown that the squeezing behaviour of the even and odd coherent states in finite dimensions is nontrivially different from those of the usual harmonic oscillator even though it approaches the latter as the dimension of the Hilbert space tends to infinity.

In this context it may be noted that finite-dimensional number states and coherent states are not merely mathematical tools but are useful in certain situations. To illustrate this [17] consider a quantum state of light ψ so that the corresponding photon number distribution function is $p_n = |\langle n|\psi\rangle|^2$. In high-$Q$ cavities [18], where cavity losses are very small, the probability p_n can be taken to be arbitrarily small beyond a threshold $n > N$ in certain situations. In this event the photon Hilbert space can be taken to be spanned by the states $|0\rangle, |1\rangle, \ldots, |N\rangle$. Theses states can be regarded as finite-dimensional number states and the coherent states constructed by the superposition of these states as the finite-dimensional coherent states. Finite dimensional coherent states can also occur in other situations as well. In a recent paper, Leonski [19] discussed generation of finite-dimensional coherent states in a system comprising a nonlinear medium of kth order driven by a weak external field. At this point we would however like to mention that this paper should be viewed as an attempt to provide a theoretical framework for finite-dimensional number states, coherent states and superpositions of the latter.

The organization of the paper is as follows. In section 2 we present the formalism concerning the finite-dimensional oscillator; in section 3 we construct the coherent states and study their nonclassical properties; in section 4 we construct the even and odd coherent states and examine their squeezing and antibunching properties; section 5 is devoted to a discussion and finally in the appendix we list some useful results used in the main body of the paper.

2. Harmonic oscillator in a finite-dimensional Hilbert space

In this section we shall briefly describe the formalism due to Atakishiev *et al* [14] concerning finite-dimensional oscillators. Let us consider a $(N+1)$ dimensional Hilbert space spanned by the states $|0\rangle, |1\rangle, \ldots, |N\rangle$ such that

$$\Phi_n^N(\xi) = \langle\xi|n\rangle = d_n^{-1} K_n^{(p)}(pN + h^{-1}\xi, N)\rho^{1/2}(pN + h^{-1}\xi) \tag{2}$$

where $0 \leqslant n \leqslant N$, $h^{-1} = \sqrt{2pqN}$, $-pN \leqslant h^{-1}\xi \leqslant qN$, $p + q = 1$. In (2) $K_n^p(x, N)$ stands for Kravchuk polynomials of degree n [20, 21] while d_n and $\rho(x)$ are the norm and the weight function [20, 21] corresponding to it. It may be noted that Kravchuk polynomials are polynomials of a discrete variable and they are orthogonal with respect to the binomial distribution as the weight function. Also for $n > N$ we have $\Phi_n^N = 0$ and orthogonality of the Kravchuk polynomials implies

$$\langle m|n\rangle = \sum_{i=0}^{N} \Phi_m^{(N)}(\xi_i)\Phi_n^{(N)}(\xi_i) = \delta_{mn}$$
$$m, n = 0, 1, 2, \ldots, N : \xi_i = (i - pN)h. \tag{3}$$

Furthermore for fixed p, if $N \to \infty$, $h \to 0$ we have [20, 21]

$$\lim_{N\to\infty} \sqrt{\frac{2}{pqN}} K_n^{(p)} \left(pN + \sqrt{2pqN}\xi\right) = \frac{1}{n!} H_n(\xi) \tag{4}$$

$$\lim_{N\to\infty} \sqrt{2pqN}\rho \left(pN + \sqrt{2pqN}\xi\right) = \frac{1}{\sqrt{\pi}} \mathrm{e}^{-\xi^2} \tag{5}$$

where $H_n(\xi)$ denotes the Hermite polynomial of degree n. Because of the properties (4) and (5) the functions $\Phi_n^{(N)}(\xi)$ become the wavefunctions of the infinite-dimensional oscillator as the dimension of the Hilbert space tends to infinity.

Let us now consider an operator $H^{(N)}$ defined by

$$H^{(N)} = 2pqN + \tfrac{1}{2} + (q-p)s - \sqrt{pq}[f(s)\mathrm{e}^{\partial_s} + \mathrm{e}^{-\partial s} f(s)] \tag{6}$$

where $f(s) = \sqrt{(qN-s)(pN+s+1)}$, $s = \frac{\xi}{h}$ and $\partial_s = \mathrm{d}/\mathrm{d}s$. Now using the recurrence relation for the Kravchuk polynomials it can be shown that

$$H^N \Phi_n^{(N)}(\xi) = (n + \tfrac{1}{2})\Phi_n^N(\xi) \qquad n = 0, 1, \ldots N \tag{7}$$

or symbolically

$$H^{(N)}|n\rangle = (n + \tfrac{1}{2})|n\rangle \qquad n = 0, 1, \ldots N. \tag{8}$$

From (8) it follows that the operator $H^{(N)}$ has a truncated spectrum consisting of $(N+1)$ eigenvalues and the eigenvalues are the same as the harmonic oscillator eigenvalues. Thus $H^{(N)}$ can be regarded as the Hamiltonian of the finite-dimensional oscillator.

Next we consider the operators A and $A^\dagger$ defined by

$$A = \sqrt{pq}[(p-q)N + 2s] + qf(s)\mathrm{e}^{\partial_s} - p\mathrm{e}^{-\partial_s} f(s) \tag{9}$$

$$A^\dagger = \sqrt{pq}[(p-q)N + 2s] + q\mathrm{e}^{-\partial_s} f(s) - pf(s)\mathrm{e}^{\partial_s} \tag{10}$$

and it can be shown that A, $A^\dagger$ and $H^{(N)}$ satisfy the following closed algebra:

$$[H^{(N)}, A] = -A \qquad [H^{(N)}, A^\dagger] = A^\dagger \qquad [A, A^\dagger] = 1 + N - 2H^{(N)}. \tag{11}$$

It is not difficult to identify the algebra in (11) as the one related to the $SU(2)$ algebra. Also using (4) and (5) it can be shown that as $N \to \infty$, equation (7) turns into the eigenvalue equation of a standard harmonic oscillator and the algebra (11) becomes, by group contraction, the Heisenberg–Weyl algebra. The action of the operators A and $A^\dagger$ on the eigenstates $\Phi_n^{(N)}(\xi)$ are given by

$$A\Phi_0^{(N)} = A^\dagger \Phi_N^{(N)} = 0 \tag{12}$$

$$A\Phi_n^{(N)} = \chi_n^{(N)} \Phi_{n-1}^{(N)} \tag{13}$$

$$A^\dagger \Phi_n^{(N)} = \chi_{n+1}^{(N)} \Phi_{n+1}^{(N)} \tag{14}$$

where $\chi_n^{(N)} = \sqrt{n(N-n+1)}$.

In terms of the kets the above relations read

$$A|0\rangle = A^\dagger|N\rangle = 0 \tag{15}$$

$$A|n\rangle = \chi_n^N |n-1\rangle \tag{16}$$

$$A^\dagger|n\rangle = \chi_{n+1}^{(N)}|n+1\rangle. \tag{17}$$

Thus A and $A^\dagger$ can be interpreted as the annihilation and creation operators and we have a finite-dimensional oscillator system which has exactly $(N+1)$ levels and which becomes a standard harmonic oscillator as $N \to \infty$. In the next section we shall construct coherent states corresponding to this system and study their properties.

3. Coherent states of the finite-dimensional oscillator and their properties

Coherent staes in finite dimensions are usually constructed following either Glauber's approach [8–10] or the truncation procedure [11, 12]. Buzek *et al* [8] used the former approach to study finite-dimensional coherent states and later analytical solutions were found by Miranowicz *et al* [9, 10]. On the other hand the truncation procedure was used by Kuang *et al* [11, 12] to construct finite-dimensional coherent states. Subsequently Opatrny *et al* [13] made a comparative study of the two procedures.

In this section we shall study nonclassical properties of finite-dimensional coherent states constructed using the operators $A^\dagger$ and A. Because of the relation (11) we can make use of the formalism of constructing spin coherent states [22–24] and the finite-dimensional coherent states are defined by

$$\begin{aligned}|\mu\rangle &= \exp(\mu A^\dagger - \mu^* A)|0\rangle \\ &= (1+|\mu|^2)^{-N/2}\exp(\mu A^+)|0\rangle \\ &= (1+|\mu|^2)^{-N/2}\sum_{n=0}^{N}\sqrt{{}^N C_n}\,\mu^n|n\rangle \end{aligned} \tag{18}$$

where μ denotes a complex number. It can be shown that the coherent states in (18) have properties such as completeness, resolution of unity etc [15, 22–24]. Also scaling A and μ by $\sqrt{N}A$ and $\frac{\mu}{\sqrt{N}}$ respectively and proceeding to the limit $N \to \infty$ we can recover the standard harmonic oscillator coherent states from (18).

It may be pointed out that difference between finite-dimensional coherent states considered here and those in [8–12] essentially lies in the choice of the weights associated with the states $|n\rangle$ in the sum in (18) (which in turn is due to the different behaviour of the states under the action of the raising and lowering operators). Also in the functional representation the coherent states in (18) can be written as

$$\psi_\mu(x) = (1+|\mu|^2)^{-\frac{N}{2}}\sum_{n=0}^{N}\sqrt{{}^N C_n}\,\mu^n\,\Phi_n^N(x) \tag{19}$$

where $\Phi_n^N(x)$ are given by (2). It may be noted that in the case of spin coherent states the r.h.s. of (19) would have contained spherical harmonics [23] instead of $\Phi_n^N(x)$.

To study the nonclassical behaviour of the coherent states we now introduce quadrature operators X and P in the following way:

$$X = \frac{A^\dagger + A}{2} \qquad P = \frac{A - A^\dagger}{2i}. \tag{20}$$

Let us now examine quadrature squeezing and antibunching properties of the coherent states. In general for any two operators A and B the squeezing condition is given by

$$S_A < 0 \qquad \text{or} \qquad S_B < 0 \tag{21}$$

where S_A is given by

$$S_A = \frac{2\langle\mu|A^2|\mu\rangle - 2\langle\mu|A|\mu\rangle^2 - |\langle\mu|[A,B]|\mu\rangle|^2}{|\langle\mu|[A,B]|\mu\rangle|^2} \tag{22}$$

and a similar definition for S_B. On the other hand the condition for the antibunching effect to take place is given by

$$g_A = \frac{\langle\mu|A^{\dagger 2}A^2|\mu\rangle}{|\langle\mu|A^\dagger A|\mu\rangle|^2} < 1. \tag{23}$$

Now parametrizing μ as $\mu = re^{i\theta}$ and using (15)–(17) and (19) we find

$$S_P < 0 \Rightarrow \frac{1 + r^4 - 2r^2\cos 2\theta}{(1 + r^2)} < |1 - r^2| \tag{24}$$

$$S_X < 0 \Rightarrow \frac{1 + r^4 + 2r^2\cos 2\theta}{(1 + r^2)} < |1 - r^2|. \tag{25}$$

From the above inequalities we find that squeezing takes place if

$$\cos 2\theta > r^2 \qquad \text{or} \qquad \cos 2\theta < -r^2 \text{ if } r < 1 \tag{26}$$

and

$$\cos 2\theta > \frac{1}{r^2} \qquad \text{or} \qquad \cos 2\theta < -\frac{1}{r^2} \text{ if } r > 1. \tag{27}$$

Clearly each of the above inequalities can be satisfied seperately and thus we conclude that the coherent states exhibit quadrature squeezing.

Let us now proceed to examine amplitude-squared squeezing. To this end we consider the following operators:

$$Y_1 = \frac{A^2 + A^{\dagger 2}}{2} \qquad Y_2 = \frac{A^2 - A^{\dagger 2}}{2}. \tag{28}$$

As before we find from (21) that

$$\begin{aligned} S_{Y_1} < 0 \Rightarrow & \frac{1}{(1+r^2)^2}[(N-2)(N-3)(r^8 + r^4 + 2r^4\cos 4\theta) - 4N(N-1)r^4\cos^2 2\theta] \\ & + \frac{2(N-2)}{(1+r^2)}[2r^2 - N(N-1)r^6] + [N(N-1)r^4 + 2] \\ & < \left| \frac{1}{(1+r^2)^2}(N-2)(N-3)(r^4 - r^8) \right. \\ & \left. + \frac{2(N-2)}{(1+r^2)}[2r^2 + (N-1)r^6] + [2 - N(N-1)r^4] \right|. \end{aligned} \tag{29}$$

Similarly

$$\begin{aligned} S_{Y_2} < 0 \Rightarrow & \frac{1}{(1+r^2)^2}[(N-2)(N-3)(r^8 + r^4 - 2r^4\cos 4\theta) - 4N(N-1)r^4\sin^2 2\theta] \\ & + \frac{2(N-2)}{(1+r^2)}[2r^2 - N(N-1)r^6] + [N(N-1)r^4 + 2] \\ & < \left| \frac{1}{(1+r^2)^2}(N-2)(N-3)(r^4 - r^8) \right. \\ & \left. + \frac{2(N-2)}{(1+r^2)}[2r^2 + (N-1)r^6] + [2 - N(N-1)r^4] \right|. \end{aligned} \tag{30}$$

Although the squeezing conditions (29) and (30) are difficult to analyse analytically for general values of N, it is still possible to analyse them analytically for some low values of N. Let us first consider the case $N = 2$. In this case, according to (29) and (30) amplitude-squared squeezing takes place if

$$\cos^2 2\theta > \frac{(1+r^2)^2}{2} \qquad \text{or} \qquad \sin^2 2\theta > \frac{(1+r^2)^2}{2} \text{ if } r < 1 \tag{31}$$

and

$$\cos^2 2\theta > \frac{(1+r^2)^2}{2r^4} \qquad \text{or} \qquad \sin^2 2\theta > \frac{(1+r^2)^2}{2r^4} \text{ if } r > 1. \tag{32}$$

Evidently the inequalities in (31) and (32) can be satisfied for many values of r and θ. Thus for $N = 2$ it is possible to have amplitude-squared squeezing.

We now consider the case $N = 4$. In this case (29) and (30) give the following conditions for squeezing:

$$\cos^2 2\theta > \frac{(r^2+5)(r^2+1)}{10} \qquad \text{or} \qquad \sin^2 2\theta > \frac{(r^2+5)(r^2+1)}{10} \text{ if } r < 1 \tag{33}$$

$$\cos^2 2\theta > \frac{(5r^2+1)(r^2+1)}{10r^4} \qquad \text{or} \qquad \sin^2 2\theta > \frac{(5r^2+1)(r^2+1)}{10r^4} \text{ if } r > 1. \tag{34}$$

It can be verified that, as in the case $N = 2$, in this case the inequalities above can also be satisfied for a number of values of r and θ. Therefore we conclude that for $N = 4$ there will also be amplitude-squared squeezing.

Finally we discuss, the antibunching effect for the coherent states. From (23) we find

$$g_2 = \frac{(N-1)[(N-1)(N+4r^2)+2r^4]}{N(N+r^2)^2}. \tag{35}$$

For $N = 2$ we obtain

$$g_2 < 1 \Rightarrow -2r^2 < 3 \tag{36}$$

which is always true and thus the coherent states exhibit antibunching effects for $N = 2$. We now consider $N = 4$. In this case we find

$$g_2 < 1 \Rightarrow r^2(r^2+2) < 14. \tag{37}$$

The inequality in (37) is satisfied for many values of r and thus for $N = 4$ the coherent states also exhibit antibunching effect.

Finally, we discuss the $N \to \infty$ limit. To examine (29), (30) and (35) in this limit it is necessary to rescale the operators A as $\sqrt{N}A$ and the complex number μ as $\mu/\sqrt{N}$. It can be shown that as $N \to \infty$ the inequalities (29) and (30) reduce to inequalities which do not hold, while (35) becomes equal to one. This implies an absence of amplitude-squared squeezing and antibunching in the infinite-dimensional limit.

4. Even and odd coherent states: their squeezing and antibunching properties

In this section we shall construct even and odd coherent states corresponding to the algebra (11) and examine their quadrature squeezing, amplitude-squared squeezing and antibunching properties. (We refer the reader to the appendix for the definition of the quantities $a_\pm^i$ and other relevant results used in this section.)

The even coherent states are defined by (henceforth we shall take $N = 2d$)

$$\begin{aligned} |\mu\rangle_e &= N_e \cosh(\mu A^\dagger)|0\rangle \\ &= N_e \sum_{n=0}^{d} \frac{\mu^{2n}}{(2n)!}(A^\dagger)^{2n}|0\rangle \\ &= N_e \sum_{n=0}^{d} \sqrt{{}^N C_{2n}}\,\mu^{2n}|2n\rangle \end{aligned} \tag{38}$$

where μ is a complex number and the normalization constant $N_{\rm e}$ is determined from the condition ${}_{\rm e}\langle\mu|\mu\rangle_{\rm e} = 1$ and is given by

$$N_{\rm e} = \left[\sum_{n=0}^{d} (\mu\bar{\mu})^{2n}\right]^{-\frac{1}{2}} = (a_+^0)^{-\frac{1}{2}}. \tag{39}$$

The odd coherent states are defined by

$$\begin{aligned}|\mu\rangle_{\rm o} &= N_{\rm o} \sinh(\mu A^\dagger)|0\rangle \\ &= N_{\rm o} \sum_{n=0}^{d-1} \sqrt{{}^N C_{2n+1}}\mu^{2n+1}|2n+1\rangle\end{aligned} \tag{40}$$

where the normalization constant is determined by the relation ${}_{\rm o}\langle\mu|\mu\rangle_{\rm o} = 1$ and is given by

$$N_{\rm o} = \left[\sum_{n=0}^{d-1} \sqrt{{}^N C_{2n+1}}(\mu\bar{\mu})^{2n}\right]^{-\frac{1}{2}} = (a_-^0)^{-\frac{1}{2}}. \tag{41}$$

We shall now examine quadrature squeezing properties of the even and odd coherent states defined in (38) and (40) respectively. From (20) and (21) it can be shown that even coherent states are squeezed if

$$[a_+^0 + 2(N-1)r^2 a_-^1 \pm 2(N-1)(r^2\cos 2\theta - r^4)a_+^2] - \tfrac{1}{2}|a_+^0 - 2r^2 a_-^1| < 0 \tag{42}$$

and odd coherent states are squeezed if

$$[a_-^0 + 2(N-1)r^2 a_+^1 \pm 2(N-1)(r^2\cos 2\theta - r^4)a_-^2] - \tfrac{1}{2}|a_-^0 - 2r^2 a_+^1| < 0. \tag{43}$$

As before we shall analyse the conditions (42) and (43) analytically for $N = 2, 4$. For $N = 2$, the even coherent states will be squeezed if

$$\cos 2\theta > r^2 \qquad \text{or} \qquad \cos 2\theta < -r^2 \text{ if } r < 1 \tag{44}$$

and

$$\cos 2\theta > \frac{1}{r^2} \qquad \text{or} \qquad \cos 2\theta < -\frac{1}{r^2} \text{ if } r > 1 \tag{45}$$

while for $N = 4$ squeezing occurs if

$$\cos 2\theta < -\frac{r^2(r^4+9)}{3(r^4+1)} \qquad \text{or} \qquad \cos 2\theta > \frac{r^2(r^4+9)}{3(r^4+1)} \text{ if } r < 1 \tag{46}$$

and

$$\cos 2\theta < -\frac{(1+9r^4)}{3r^2(1+r^4)} \qquad \text{or} \qquad \cos 2\theta > \frac{(1+9r^4)}{3r^2(1+r^4)} \text{ if } r > 1. \tag{47}$$

There are many values of r and θ for which the inequalities in (44)–(47) are satisfied and thus the even coherent states exhibit quadrature squeezing for both $N = 2, 4$.

Let us now consider the odd coherent states. For $N = 2$, the squeezing conditions (43) reduce to

$$r^2 < 0 \tag{48}$$

and for $N = 4$ these conditions are

$$\cos 2\theta < -\frac{(2+3r^4)}{3r^2} \qquad \text{or} \qquad \cos 2\theta > \frac{(2+3r^4)}{3r^2} \text{ if } r < 1 \tag{49}$$

and

$$\cos 2\theta < -\frac{(3+2r^4)}{3r^2} \qquad \text{or} \qquad \cos 2\theta > \frac{(3+2r^4)}{3r^2} \text{ if } r > 1. \tag{50}$$

It can be verified that that the inequalities in (48)–(50) are not satisfied for any r and θ. Thus the odd coherent states do not exhibit quadrature squeezing for $N = 2, 4$.

Now we consider the limit $N \to \infty$. Using the same scaling as mentioned in section 3, it can be shown that in the limit $N \to \infty$, (42) becomes

$$\cos 2\theta < -\tanh r^2 \qquad \text{or} \qquad \cos 2\theta > \tanh r^2 \tag{51}$$

while (43) becomes

$$\cos 2\theta < -\coth r^2 \qquad \text{or} \qquad \cos 2\theta > \coth r^2. \tag{52}$$

Thus as $N \to \infty$, even coherent states exhibit quadrature squeezing while odd coherent states do not show this effect [25, 26]. Next we examine whether or not the even/odd coherent states exhibit amplitude-squared squeezing. From (21) we find that the even coherent states will exhibit amplitude-squared squeezing if

$$\begin{aligned}(N-2)(N-3)a_+^4(r^8 + 2r^4\cos 4\theta + r^4) + 2(N-2)a_-^3[2r^2 - (N-1)r^6]&\\ +a_+^2[2 + N(N-1)r^4] - 4N_e^2 N(N-1)a_+^2 r^4\cos^2 2\theta&\\ < |(N-2)(N-3)a_+^4(r^4 - r^8) + 2(N-2)[2r^2 + (N-1)r^6]a_-^3&\\ +a_+^2[2 - N(N-1)r^4]|&\end{aligned} \tag{53}$$

or

$$\begin{aligned}(N-2)(N-3)a_+^4(r^8 - 2r^4\cos 4\theta + r^4) + 2(N-2)a_-^3[2r^2 - (N-1)r^6]&\\ +a_+^2[2 + N(N-1)r^4] - 4N_e^2 N(N-1)a_+^2 r^4\sin^2 2\theta&\\ < |(N-2)(N-3)a_+^4(r^4 - r^8) + 2(N-2)[2r^2 + (N-1)r^6]a_-^3&\\ +a_+^2[2 - N(N-1)r^4]|.&\end{aligned} \tag{54}$$

For $N = 2$ the above conditions become

$$\cos^2 2\theta > \frac{1+r^4}{2} \qquad \text{or} \qquad \sin^2 2\theta > \frac{1+r^4}{2} \ \text{ if } r < 1 \tag{55}$$

and

$$\cos^2 2\theta > \frac{1+r^4}{2r^4} \qquad \text{or} \qquad \sin^2 2\theta > \frac{1+r^4}{2r^4} \ \text{ if } r > 1. \tag{56}$$

Similarly for $N = 4$ we find

$$\cos^2 2\theta > \frac{r^{12} + 11r^8 + 31r^4 + 5}{2(5r^8 + 6r^4 + 5)} \qquad \text{or} \qquad \sin^2 2\theta > \frac{r^{12} + 11r^8 + 31r^4 + 5}{2(5r^8 + 6r^4 + 5)} \quad \text{if } r < 1 \tag{57}$$

and

$$\cos^2 2\theta > \frac{5r^{12} + 31r^8 + 11r^4 + 1}{2r^4(5r^8 + 6r^4 + 5)} \qquad \text{or} \qquad \sin^2 2\theta > \frac{r^{12} + 11r^8 + 31r^4 + 5}{2(5r^8 + 6r^4 + 5)} \quad \text{if } r > 1. \tag{58}$$

Since the conditions in (55)–(58) can be satisfied for many values of θ and r we conclude that the even coherent states are amplitude-squared squeezed for $N = 2, 4$.

We now turn to the odd coherent states. The conditions for amplitude-squared squeezing follows from (27) and are given by

$$\begin{aligned}(N-2)(N-3)a_-^4(r^8 + 2r^4\cos 4\theta + r^4) + 2(N-2)a_-^3[r^2 - (N-1)r^6]&\\ +N(N-1)r^4 a_-^2 - 4N_e^2 N(N-1)a_-^2 2r^4\cos^2 2\theta&\\ < |(N-2)(N-3)a_-^4(r^4 - r^8) + 2(N-2)[r^2 - (N-1)r^6]a_+^3&\\ -N(N-1)r^4 a_-^2|&\end{aligned} \tag{59}$$

or

$$\begin{aligned}(N-2)(N-3)a_-^4(r^8-2r^4\cos 4\theta+r^4)+2(N-2)a_+^3[r^2-(N-1)r^6]\\ +N(N-1)r^4a_+^2-4N_o^N(N-1)a_-^2r^4\sin^2 2\theta\\ <|(N-2)(N-3)a_-^4(r^4-r^8)+2(N-2)[r^2-(N-1)r^6]a_+^3\\ -N(N-1)r^4a_-^2|.\end{aligned} \tag{60}$$

It can be shown that while for $N=2$ there is no amplitude-squared squeezing, the conditions for this to take place for $N=4$ are given by

$$\cos^2 2\theta > \frac{1+r^4}{2} \qquad \text{or} \qquad \sin^2 2\theta > \frac{1+r^4}{2} \text{ if } r<1 \tag{61}$$

and

$$\cos^2 2\theta > \frac{1+r^4}{6r^4} \qquad \text{or} \qquad \sin^2 2\theta > \frac{1+r^4}{6r^4} \text{ if } r>1. \tag{62}$$

Since the inequalities in (61) and (62) can be satisfied for many values of r and θ, we conclude that odd coherent states exhibit amplitude-squared squeezing for $N=4$. Finally we discuss the antibunching effect for the even and odd coherent states. From (28) we find

$$g_e=\frac{(N-1)^2r^4a_+^2-2(N-1)^2(1-\frac{2}{N})r^6a_-^3+(N-1)(1-\frac{2}{N})(N-3)r^8a_+^4}{N_e^2|a_+^0+(N-2)r^2a_-^1-(N-1)r^4a_+^2|^2} \tag{63}$$

$$g_o=\frac{(N-1)^2r^4a_-^1-2(N-1)^2(1-\frac{2}{N})r^4a_+^3+(N-1)(1-\frac{2}{N})(N-3)r^6a_-^4}{N_o^2|Na_+^1-(N-1)r^2a_-^2|^2}. \tag{64}$$

From (62) and (63) it can be shown that for $N=2$ the even coherent states do not show antibunching while the odd coherent states show this effect. On the other hand for $N=4$, both the even and the odd coherent states exhibit the antibunching effect.

Let us now discuss the limit $N\to\infty$. It can be verified that as $N\to\infty$ the r.h.s. and l.h.s. of (53), (54) as well as of (59) and (60) become equal so that amplitude-squared squeezing does not take place for either the even coherent states or the odd coherent states [25, 26]. Also it can be shown that

$$\begin{aligned} g_e&=\coth r^2>1 \qquad \text{as } N\to\infty\\ g_o&=\tanh r^2<1 \qquad \text{as } N\to\infty.\end{aligned} \tag{65}$$

Thus in the infinite-dimensional limit, the odd coherent states show antibunching effects while the even coherent states do not [26].

From the results obtained here it is clear that the behaviour of the even and odd coherent states in finite dimensions is quite different from those in infinite dimensions. For instance the even coherent states exhibit amplitude-squared squeezing in finite dimensions but not so in infinite dimensions. To understand the situation at a qualitative level we refer to the work of Buzek *et al* [27]. From [27] we conclude that in this case nonclassical behaviour shown by the even and odd coherent states is due to the quantum interference between different coherent components of the superposition states. It may be noted that the quantum interference between the components also depends on the dimension N of the space. In some of the cases nonclassical effects which are present at finite values of N disappear as $N\to\infty$. In other words the quantum interference vanishes in the limit $N\to\infty$.

5. Discussion

In this paper we have considered a harmonic oscillator in a finite-dimensional Hilbert space and studied coherent states, even coherent states as well as odd coherent states corresponding to this system. The oscillator system considered here has a closed symmetry algebra related to $SU(2)$ algebra and the states of the oscillator are represented by Kravchuk polynomials. Thus in contrast to [1–4], in this case the states of the oscillator exist in a finite-dimensional space. Within this framework, we have studied various properties of the coherent states, even and odd coherent states. It has been shown that the nonclassical properties (e.g. squeezing, antibunching) of these states are rather different from the corresponding states of the infinite-dimensional oscillator. In particular, we have obtained the following results: (1) coherent states exhibit quadrature squeezing, amplitude-squared squeezing as well as antibunching, (2) even coherent states exhibit quadrature and amplitude-squared squeezing but no antibunching, (3) odd coherent states do not exhibit quadrature squeezing but they show amplitude-squared squeezing for $N = 4$ and also antibunching. It has also been shown that the finite-dimensional coherent states as well as the even and odd coherent states have the correct limiting behaviour as $N \to \infty$. It may be noted that recently nonclassical states of the harmonic oscillator have been engineered in experiments with trapped ions [28]. We hope that finite-dimensional counterparts of these states will be found in some future experiments.

Acknowledgments

The authors thank the referees for constructive criticisms and suggestions. PR thanks Ph Feinsilver for useful communications.

Appendix

Here we shall list some results concerning the expectation values of various operators. First we define

$$a_\pm^i = \frac{(1+r^2)^{N-i} \pm (1-r^2)^{N-i}}{2} \qquad i = 0, 1, 2, 3, 4. \tag{A1}$$

Then

$$_{\rm e,o}\langle\mu|A^2|\mu\rangle_{\rm e,o} = N_{\rm e,o}^2 \mu^2 N(N-1) a_\pm^2 \tag{A2}$$

$$_{\rm e,o}\langle\mu|A^{\dagger 2}|\mu\rangle_{\rm e,o} = N_{\rm e,o}^2 \bar{\mu}^2 N(N-1) a_\pm^2 \tag{A3}$$

$$_{\rm e,o}\langle\mu|AA^\dagger|\mu\rangle_{\rm e,o} = N_{\rm e,o}^2 [N a_\pm^0 + N(N-2)(\mu\bar{\mu}) a_\mp^1 - N(N-1)(\mu\bar{\mu})^2 a_\pm^2] \tag{A4}$$

$$_{\rm e,o}\langle\mu|A^\dagger A|\mu\rangle_{\rm e,o} = N_{\rm e,o}^2 [N^2(\mu\bar{\mu}) a_\mp^1 - N(N-1)(\mu\bar{\mu})^2 a_\pm^2] \tag{A5}$$

$$_{\rm e,o}\langle\mu|A^4|\mu\rangle_{\rm e,o} = N_{\rm e,o}^2 \prod_{j=0}^{3} (N-j) \mu^4 a_\pm^4 \tag{A6}$$

$$_{\rm e,o}\langle\mu|A^{\dagger 4}|\mu\rangle_{\rm e,o} = N_{\rm e,o}^2 \prod_{j=0}^{3} (N-j) \bar{\mu}^4 a_\pm^4 \tag{A7}$$

$$\begin{aligned} _{\rm e}\langle\mu|A^2 A^{\dagger 2}|\mu\rangle_{\rm e} = N_{\rm e}^2 [&2N a_\mp^2 + 4N(N-1)(N-2)(\mu\bar{\mu}) a_-^3 \\ &+ N(N-1)(N-2)(N-3)(\mu\bar{\mu})^2 a_+^4] \end{aligned} \tag{A8}$$

$$_{\rm e}\langle\mu|A^2 A^{\dagger 2}|\mu\rangle_{\rm e} = N_{\rm o}^2 [2N(N-1)(N-2)(\mu\bar{\mu}) a_+^3 + N(N-1)(N-2)(N-3)(\mu\bar{\mu})^4 a_-^4] \tag{A9}$$

$$\begin{aligned} _{\rm e,o}\langle\mu|A^{\dagger 2} A^2|\mu\rangle_{\rm e,o} = N_{\rm e,o}^2 [&N^2(N-1)^2(\mu\bar{\mu})^2 a_\pm^2 - 2N(N-1)^2(N-2)(\mu\bar{\mu})^3 a_\mp^3 \\ &+ N(N-1)(N-2)(N-3)(\mu\bar{\mu})^4 a_\pm^4]. \end{aligned} \tag{A10}$$

References

[1] Pegg D T and Barnett S M 1988 *Europhys. Lett.* **6** 483
[2] Barnett S M and Pegg D T 1989 *J. Mod. Opt.* **36** 7
[3] Pegg D T and Barnett S M 1989 *Phys. Rev.* A **39** 1665
[4] Barnett S M and Pegg D T 1992 *J. Mod. Opt.* **39** 2121
[5] Ekert A K 1991 *Phys. Rev. Lett.* **67** 661
Collins G P 1992 *Phys. Today* **45** 23
Bennett C H, Brassand G and Mermin N D 1992 *Phys. Rev. Lett.* **68** 557
[6] Bennett C H, Brassand G, Crepean C, Jozsa R, Peres A and Wootters W K 1993 *Phys. Rev. Lett.* **70** 1895
Bennett C H and Wiesner S J 1992 *Phys. Rev. Lett.* **69** 2881
[7] Deutsch D 1985 *Proc. R. Soc.* A **400** 97
Landauer R 1991 *Phys. Today* **44** 23
Chuang I L and Yamamoto Y 1995 *Phys. Rev*. **52** 3489
Barranco A V *et al* 1995 *Phys. Rev.* A **52** 3457
[8] Buzek V, Wilson-Gordon A D, Knight P L and Lai W K 1992 *Phys. Rev.* A **45** 8079 and references cited therein
[9] Miranowicz A, Piatek K and Tanas R 1994 *Phys. Rev.* A **50** 3423
[10] Miranowicz A, Opatrny T and Bajer J 1996 *Concepts and Advances in Quantum Optics and Spectroscopy of Solids* ed T Hauioglv and A S Shumovsky (Dordrecht: Kluwer)
[11] Le-Man Kuang, Fa-Bo Wang and Yan-Guo Zhou 1993 *Phys. Lett.* A **183** 1
[12] Kuang L M and Chen X 1994 *Phys. Rev.* A **50** 4228
Kuang L M and Chen X 1994 *Phys. Lett.* A **186** 8
Kuang L M and Zhou J Y 1996 *J. Phys. A: Math. Gen.* **29** 895
[13] Opatrny T, Miranowicz A and Bajer J 1996 *J. Mod. Opt.* **43** 417
[14] Atakishiev N M and Suslov S K 1991 *Theor. Math. Phys.* **85** 1055
[15] Perelomov A M 1972 *Commun. Math. Phys.* **26** 222
[16] Zhu J Y and Kuang L M 1994 *Phys. Lett.* A **193** 227
[17] Figurny P, Orlowski A and Wodkiewicz K 1993 *Phys. Rev.* A **47** 5151
[18] Hinds E A 1990 *Adv. At. Mol. Phys.* **28** 237
[19] Leonski W 1997 *Phys. Rev.* A **55** 3874
Leonski W 1995 *Acta Phys. Slov.* **45** 383
Also see Vogel K, Akulin V M and Schleich W P 1993 *Phys. Rev. Lett.* **71** 1816
Parkins A S *et al* 1993 *Phys. Rev. Lett.* **71** 3095
Leonski W and Tanas R 1994 *Phys. Rev.* A **49** R20
[20] Nikiforov A F and Suslov V B 1988 *Special Functions of Mathematical Physics* (Boston, MA: Birkhauser)
[21] Szego G 1976 *Orthogonal Polynomials* (New York: American Mathematical Society)
[22] Radcliffe J M 1971 *J. Phys. A: Math. Gen.* **4** 313
[23] Arecchi F T, Courtens E, Gilmore R and Thomas H 1972 *Phys. Rev.* A **6** 2211
[24] Wodkiewicz K and Eberly J H 1985 *J. Opt. Soc. Am.* B **2** 458
[25] Hillery M 1987 *Phys. Rev.* A **36** 3796
[26] Xia Y and Guo G 1989 *Phys. Lett.* A **136** 281
[27] Buzek V, Vidiella-Barranco A and Knight P L 1992 *Phys. Rev.* **45** 6570
Buzek V and Knight P L 1995 *Prog. Opt.* **34** 1 and references therein
[28] Leibfried *et al* 1996 *Phys. Rev. Lett.* **77** 4281